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Magnus force and Hellmann–Feynman force: path integral approach

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Abstract

This paper considers the derivation of the Magnus force from a model system consisting of a single vortex imbedded in a uniform positive background coupled with a mutual interaction charged boson. By eliminating the charged boson degree of freedom, the effective action of a single vortex is obtained and can be used to derive the Hellmann–Feynman force. From the ground state contribution a Magnus force is obtained.

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The Magnus force or lift force of classical hydrodynamics arises as a consequence of its motion through the fluid. The argument for the existence of a Magnus force on a vortex line in the type II superconductor was first proposed by Friedel *et al* [1] and later developed and extended by Nozieres and Vinen [2] by including pinning and friction. It was believed that the existence of a vortex is a general property of the system. In this paper we show that the existence of the Magnus force is a general property of a vortex. We propose a microscopic derivation of the Magnus force from a model system consisting of a single vortex coupled to mutual-interacting charged bosons and imbedded in a uniform positive background. By eliminating the charged boson degree of freedom an effective Lagrangian is obtained containing the generalized Hellmann–Feynman force which can be used to derive the Magnus force. This force can be obtained by considering that the ground state contribution leads to the Magnus force.

The full Hamiltonian for a quantized vortex coupled to an interacting charged boson imbedded in a positive uniform background is given as

$$\hat{H} = \hat{H}_v + \hat{h}_c + \hat{h}. \tag{1}$$

Here, $H_v(\hat{\vec{P}}, \hat{\vec{R}})$ is the Hamitonian for a quantized vortex in which $\vec{R}$ denotes the position of the vortex and $\vec{P}$ its conjugate momentum. The second term

$$\hat{h}_c = -\frac{e^2}{2m^2c} \sum_{n \neq n'} p_n^i T^{ij}(\vec{x}_n - \vec{x}_{n'}) p_{n'}^j \tag{2}$$

with $T^{ij}(\vec{x}) = (\delta^{ij}|\vec{x}|^{-1} + x^i x^j|\vec{x}|^{-3})/2c$, the current–current interaction, and m is the boson mass. The Hamiltonian $\hat{h}_c$ represents the lowest-order relativistic effects, an interaction first obtained by Darwin in 1920 [3]. The last term

$$\hat{h} = \sum_n \frac{\left(\hat{\vec{p}}_n - \frac{2e}{c}\hat{\vec{a}}(\hat{\vec{x}}_n - \hat{\vec{R}})\right)^2}{2m} + \frac{1}{2}\sum_{n\neq n'} U(\hat{\vec{x}}_n - \hat{\vec{x}}_{n'}) + \hat{h}_b \tag{3}$$

is the Hamiltonian representing N bosons with negative charge $-e$, interacting with the vector potential $\vec{a}(\hat{\vec{x}}_n - \hat{\vec{R}})$ and satisfying the equation $\oint \vec{a}(\vec{x}_n - \vec{R}) \cdot \mathrm{d}\vec{l} = \phi_0 = hc/2e$. The $U(\hat{\vec{x}}_n - \hat{\vec{x}}_{n'})$ term represents the mutual Coulomb interaction. Finally $\hat{h}_b$ is the uniform positive background and

$$\hat{h}_b = -\sum_n \int \mathrm{d}^3\vec{x}' e^2 \bar{n}(\vec{x}')|\vec{x}_n - \vec{x}'|^{-1} \tag{4}$$

where $\bar{n}(\vec{x}')$ is the charge distribution of the lattice and accounts for the interaction with the uniform positive background charge $e\bar{n}$.

Next, the full Hamiltonian $\hat{H}$ can be separated into two parts—the internal and the collective. The internal part, $\hat{h}_i = \hat{h} + \hat{h}_c$, is dependent on the centre point of the vortex, $\vec{R}$, and not explicitly on the conjugate momentum of the vortex, $\vec{P}$. The collective Hamiltonian, $\hat{H}_v$, is the Hamiltonian for a quantized vortex.

In considering the probability amplitude for a quantum process starting from the initial position, $\vec{x}_{a1}, \ldots, \vec{x}_{aN}$, $\vec{R}_a$ at t_a, and returning to the final position $\vec{x}_{b1}, \ldots, \vec{x}_{bN}$, $\vec{R}_b$ at t_b, the propagator can be written as

$$K(\vec{x}_{1b}, \ldots, \vec{x}_{Nb}, \vec{R}_b, t_b; \vec{x}_{1a}, \ldots, \vec{x}_{Na}, \vec{R}_a, t_a) = \sum_m \sum_n \Psi_m(\vec{x}_{1b}, \ldots, \vec{x}_{Nb}; \vec{R}_b)$$
$$\times \langle m; \vec{R}_b|\langle \vec{R}_b| \exp\left[-\frac{\mathrm{i}}{\hbar}\hat{H}(t_b - t_a)\right]|\vec{R}_a\rangle|n; \vec{R}_a\rangle \Psi_n^*(\vec{x}_{1a}, \ldots, \vec{x}_{Na}; \vec{R}_a). \tag{5}$$

By inserting complete sets of coordinate states and a complete set of momentum states at $t = t_\mathrm{k}$, with $\varepsilon = \frac{t_b - t_a}{L}$ it is possible to consider the following relationship as $\varepsilon \to 0$

$$\langle \vec{R}_k| \exp\left[-\frac{\mathrm{i}}{\hbar}\hat{H}\varepsilon\right]|\vec{R}_{k-1}\rangle \approx \langle \vec{R}_k| \exp\left[-\frac{\mathrm{i}}{\hbar}\vec{H}_v(\hat{\vec{P}}, \hat{\vec{R}})\varepsilon\right]|\vec{R}_{k-1}\rangle \exp\left[\frac{\mathrm{i}}{\hbar}\hat{h}_i(\hat{\vec{x}}, \hat{\vec{p}}; \vec{R}_k)\varepsilon\right]$$
$$= \int \mathrm{d}\vec{P}_k \exp\left[\frac{\mathrm{i}}{\hbar}\varepsilon\left[\vec{P}_k \cdot \left(\frac{\vec{R}_k - \vec{R}_{k-1}}{\varepsilon}\right) - \hat{H}_v(\hat{\vec{P}}, \hat{\vec{R}})\right]\right] \exp\left[\frac{\mathrm{i}}{\hbar}\hat{h}_i(\hat{\vec{x}}, \hat{\vec{p}}; \vec{R}_k)\varepsilon\right]. \tag{6}$$

Then equation (5) can be expressed as

$$K(\vec{x}_{1b}, \ldots, \vec{x}_{Nb}, \vec{R}_b, t_b; \vec{x}_{1a}, \ldots, \vec{x}_{Na}, \vec{R}_a, t_a)$$
$$= \sum_m \sum_n \Psi_m(\vec{x}_{1b}, \ldots, \vec{x}_{Nb}; \vec{R}_b)\Psi_n^*(\vec{x}_{1a}, \ldots, \vec{x}_{Na}; \vec{R}_a)$$
$$\times \int D[\vec{P}]D[\vec{R}]T_{mn} \exp\left[-\frac{\mathrm{i}}{\hbar}S[\vec{R}(t), \vec{P}(t)]\right] \tag{7}$$

where $S[\vec{R}(t), \vec{P}(t)] = \int_{t_a}^{t_b}[\vec{P} \cdot \dot{\vec{R}} - H_v(\vec{P}, \vec{R})]\mathrm{d}t$ is the action of the collective motion along the path between a and b. Here, $\Psi_n(\vec{x}_{1a}, \ldots, \vec{x}_{Na}; \vec{R}_a)(\Psi_m(\vec{x}_{1b}, \ldots, \vec{x}_{Nb}; \vec{R}_b))$ are the wavefunctions of the internal part, $\hat{h}_i = \hat{h} + \hat{h}_c$ at $\vec{R} = \vec{R}_a(\vec{R}_b)$ with eigenvalue $E_n(\vec{R}_a)(E_m(\vec{R}_b))$ and the external variable $\vec{R} = \vec{R}_a(\vec{R}_b)$. T_{mn} is just the transition amplitude

between the quantum states from $\Psi_m(\vec{x}_{1b},\ldots,\vec{x}_{Nb};\vec{R}_b)$ to $\Psi_n(\vec{x}_{1a},\ldots,\vec{x}_{Na};\vec{R}_a)$ and is given by

$$T_{mn} = \langle m;\vec{R}_b|\exp\left[-\frac{\mathrm{i}}{\hbar}\hat{h}_i(\vec{R}(t_b))\varepsilon\right]\cdots\exp\left[-\frac{\mathrm{i}}{\hbar}\hat{h}_i(\vec{R}(t_a))\varepsilon\right]|n;\vec{R}_a\rangle. \tag{8}$$

By inserting the completeness relationship holding for the internal state $\hat{h}_i$ at each point of the external variable $\vec{R}_k$, $\sum_{j_k}|j_k;\vec{R}_k\rangle\langle j_k;\vec{R}_k| = 1$, equation (8) can be written as

$$T_{mn} = \sum_{j_1}\cdots\sum_{j_L}\langle m;\vec{R}_b|\exp\left[-\frac{\mathrm{i}}{\hbar}\hat{h}_i(\vec{R}(t_b))\varepsilon\right]|j_L;\vec{R}_L\rangle\cdots$$
$$\times\langle j_1;\vec{R}_1|\exp\left[-\frac{\mathrm{i}}{\hbar}\hat{h}_i(\vec{R}(t_a))\varepsilon\right]|n;\vec{R}_a\rangle. \tag{9}$$

In the adiabatic approximation, an example is Berry's 1985 phase [4], the quantum transition between states with the same quantum number n only is picked up and is described by the matrix element $\langle n;\vec{R}_{k+1}|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{h}_i(\vec{R}_k)\varepsilon}|n;\vec{R}_k\rangle$. Thus by using the approximate relation

$$\langle n;\vec{R}_{k+1}|\exp\left[\frac{\mathrm{i}}{\hbar}\hat{h}_i(\vec{R}_k)\varepsilon\right]|n;\vec{R}_k\rangle \approx [1-\langle n;\vec{R}|\vec{\nabla}_R|n;\vec{R}\rangle\cdot\dot{\vec{R}}\varepsilon]\exp\left[-\frac{\mathrm{i}}{\hbar}\varepsilon E_n(\vec{R}_k)\right]$$
$$= \exp\left[\frac{\mathrm{i}}{\hbar}\varepsilon(-E_n(\vec{R}_k)+\mathrm{i}\hbar\vec{A}_{n,n}\cdot\dot{\vec{R}})\right] \tag{10}$$

equation (8) becomes

$$T_{mn} = \delta_{m,n}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b}(E_n(\vec{R})-\mathrm{i}\hbar\vec{A}_{n,n}\cdot\dot{\vec{R}})\mathrm{d}t\right] \tag{11}$$

where

$$\vec{A}_{n,n} = \langle n;\vec{R}|\vec{\nabla}_R|n;\vec{R}\rangle. \tag{12}$$

The vector potenial $\vec{A}_{n,n}$ implies the property of the internal part of the Hamiltonian $\hat{h}_i$ in the form of ket vector $|n;\vec{R}\rangle$. We arrive at the effective path integral associated with the adiabatic approximation of the dynamical variable $\vec{R}$,

$$K(\vec{x}_{1b},\ldots,\vec{x}_{Nb},\vec{R}_b,t_b;\vec{x}_{1a},\ldots,\vec{x}_{Na},\vec{R}_a,t_a)$$
$$= \sum_m\sum_n\Psi_m(\vec{x}_{1b},\ldots,\vec{x}_{Nb};\vec{R}_b)\Psi_n^*(\vec{x}_{1a},\ldots,\vec{x}_{Na};\vec{R}_a)K_{mn} \tag{13}$$

K_{mn} gives the usual dynamical evolution of the wavefunction of the internal part with an additional effect from the motion of the external variable over all possible paths. Therefore, the evolution kernel K_{mn} can be expressed as,

$$K_{mn} = \delta_{m,n}\int D[\vec{P}]D[\vec{R}]\exp\left[\frac{\mathrm{i}}{\hbar}\left[\int_{t_a}^{t_b}\mathrm{d}t([\vec{P}\cdot\dot{\vec{R}}-H_v]-E_m(\vec{R})+\mathrm{i}\hbar\vec{A}_{n,n}\cdot\dot{\vec{R}})\right]\right] \tag{14}$$

where, $L_{n,n}^{\mathrm{eff}} = [\vec{P}\cdot\dot{\vec{R}}-H_v]-E_n(\vec{R})+\mathrm{i}\hbar\vec{A}_{n,n}\cdot\dot{\vec{R}}$ is the effective Lagrangian corresponding with Schrödinger's equation for molecular physics given by the Born–Oppenheimer approximation [5], a matrix-valued Schrödinger operator for the nuclear wavefunction. If the external variable $\vec{R}(t)$ is to describe an adiabatic motion returning via a closed path C then the third term in the exponent of equation (14) is immediately recognized as Berry's 1985 phase [4]:

$$\Gamma_n = \mathrm{i}\hbar\oint_C\langle n;\vec{R}|\vec{\nabla}_R|n;\vec{R}\rangle\cdot\mathrm{d}\vec{R}. \tag{15}$$

To obtain the Hellmann–Feynman force, we define the force on the vortex from the Lagrange equation

$$\frac{\partial}{\partial\vec{R}}L_m^{\mathrm{eff}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{\vec{R}}}L_m^{\mathrm{eff}} = 0. \tag{16}$$

Then the new force, which is in addition to the original force, can be written as

$$F_m^X = \mathrm{i}\hbar\dot{R}^X\left[\left\langle\frac{\partial\Psi_m}{\partial Y}|\frac{\partial\Psi_m}{\partial X}\right\rangle - \left\langle\frac{\partial\Psi_m}{\partial X}|\frac{\partial\Psi_m}{\partial Y}\right\rangle\right] - \frac{\partial}{\partial X}E_m(\vec{R}) \tag{17}$$

$$F_m^Y = \mathrm{i}\hbar\dot{R}^Y\left[\left\langle\frac{\partial\Psi_m}{\partial Y}|\frac{\partial\Psi_m}{\partial X}\right\rangle - \left\langle\frac{\partial\Psi_m}{\partial X}|\frac{\partial\Psi_m}{\partial Y}\right\rangle\right] - \frac{\partial}{\partial Y}E_m(\vec{R}). \tag{18}$$

The above result can be easily recognized as the Hellmann–Feynmen theorem [6]. Next the Magnus force can be derived by using the many-body wavefunction proposed by Ao and Thouless [7]. This wavefunction contains both amplitude and phase varying in space and time

$$\Psi_0(\vec{x}_1,\ldots,\vec{x}_N;\vec{R}) = \tilde{\Psi}_0(\vec{x}_1,\ldots,\vec{x}_N;\vec{R})\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\Theta(\vec{x}_j-\vec{R})\right] \tag{19}$$

where $\tilde{\Psi}_0$, in the absence of the external magnetic field, is the many-body wavefunction of a superconductor. The ground state wavefunction depends on the positions of the N bosons in the system. Since the wavefunction can be determined in such a way that the dependence on $\vec{x}$ is entirely through $\vec{x}-\vec{R}$, the partial derivatives with respect to $\vec{R}$ can be replaced by a sum over partial derivatives with respect to the particle coordinate $\vec{x}_i$. The probability of any particular configuration is proportional to $|\tilde{\Psi}_0|^2$, with the normalization,

$$\int\cdots\int|\tilde{\Psi}_0|^2\mathrm{d}^2\vec{x}_1\cdots\mathrm{d}^2\vec{x}_N = N \tag{20}$$

and

$$\int\cdots\int|\tilde{\Psi}_0|^2\mathrm{d}^2\vec{x}_1\cdots\mathrm{d}^2\vec{x}_{N-1} = \rho(\vec{x},\vec{R}) \tag{21}$$

where $\rho(\vec{x},\vec{R})$ is the probability density. The probability density $\rho(\vec{x},\vec{R})$ must satisfy the boundary conditions; that is, the density $\rho(\vec{x},\vec{R})$ must vanish continuously at $\vec{x}=\vec{R}$ as well as approach the background density ρ_0 as $|\vec{x}-\vec{R}|\to\infty$. Therefore, the Magnus force from the first term in equations (17) and (18) can be defined as,

$$\vec{F}_{\mathrm{Magnus}} = \dot{\vec{R}}\times\mathrm{i}\hbar\vec{\nabla}_R\times\langle\Psi_0;\vec{R}|\vec{\nabla}_R|\Psi_0;\vec{R}\rangle. \tag{22}$$

By virtue of the property of the many-body wavefunction and the ground state condition, the Magnus force becomes

$$\vec{F}_{\mathrm{Magnus}} = \dot{\vec{R}}\times\vec{\nabla}_R\times\int\mathrm{d}^2\vec{x}N\rho(\vec{x},\vec{R})\vec{\nabla}_R\Theta(\vec{x}-\vec{R}). \tag{23}$$

Using Stokes theorem and the relation

$$\vec{\nabla}_R\Theta(\vec{x}-\vec{R}) = \frac{\hat{k}\times(\vec{x}-\vec{R})}{|\vec{x}-\vec{R}|^2} \tag{24}$$

the following equation is finally obtained:

$$\vec{F}_{\mathrm{Magnus}} = 2\pi\rho_s\hbar\hat{k}\times\dot{\vec{R}} \tag{25}$$

where $\rho_s = N\rho_0$ is the number density.

Thus, a force is exerted on the vortex when it moves relative to the fluid density. This Magnus force is proportional to and perpendicular to the vortex velocity, and proportional to the fluid density. The Magnus force makes the vortex dynamics similar to that of charged particles in a magnetic field, with the role of the magnetic field played by the fluid density. This problem is discussed in our previous papers [8, 9]. However, it is interesting to point out that in this formulation, the mass of the vortex in the canonical momentum $\vec{P}$ was deliberately

hidden. The mass of the vortex is still controversial; this is addressed in another paper [10]. In conclusion we have demonstrated that the origin of the Magnus force is an effect of the transition amplitude of the supercurrent and is independent of the mutual interaction of the boson. The quantum transition between states is a result of interaction between the vector potenial of a vortex and a supercurrent or charged boson. The existence of the Magnus force in a neutral fluid is an effect of pressure. This is the difference between the Magnus force in a superconductor and that in a neutral fluid. These findings support the belief that the Magnus force is a general property of the system.

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